Numerical Methods, Existence and Uniqueness for the Thermoelastics Systems with Moving Boundary

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Abstract

In this work, we are interested in obtaining the existence, uniqueness of the solution and an approximated numerical solution for the model of linear thermoelasticity with moving boundary. We apply the finite element method with a finite difference method to obtain an approximated numerical solution. Some numerical experiments were presented to show the moving boundary's effects in the problems in linear thermoelasticity.

Keywords: Thermoelasticity system; Moving Boundary; Finite Element Method; Finite Difference Method.

1 Introduction

Let $Q_t = \{(x,t) \in I\!R^2;\, \alpha(t) < x < \beta(t),\, 0 < t < T\}$ be the non-cylindrical domain with boundary $\Sigma_t = \bigcup_{0<t<T} \{\alpha(t), \beta(t)\} \times \{t\}$ and consider the following problem:

$$
\textbf{(I)} \qquad \left[\begin{array}{ll}
\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial^2 u}{\partial x^2} + \eta_1 \dfrac{\partial \theta}{\partial x} = 0, & \forall\, (x,t) \in Q_t \\[2mm]
\dfrac{\partial \theta}{\partial t} - k \dfrac{\partial^2 \theta}{\partial x^2} + \eta_2 \dfrac{\partial^2 u}{\partial x \partial t} = 0, & \forall\, (x,t) \in Q_t, \\[2mm]
u = \theta = 0, & \forall\, (x,t) \in \Sigma_t, \\[2mm]
u(x,0) = u_0(x),\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x); & \alpha(0) < x < \beta(0).
\end{array} \right.
$$

Existence and uniqueness of the elasticity linear and nonlinear inside bounded and unbounded cylindrical domains, have been studied by several authors, among them, [3] and [4].

In this work, we will investigate the existence, uniqueness and approximated solution of the problem (**I**). We also will show the influence of moving boundary employing numerical examples. For this we consider the following hypotheses:

H1: $\alpha, \beta \in C^2([0,T); I\!R)$,

with $0 < \gamma_0 = \min_{0 \le t \le T} \gamma(t), \quad \text{where } \gamma(t) = \beta(t) - \alpha(t),$

H2: $\exists k_1 \in I\!R$, such that,

$0 < k_1 < 1 - (\alpha'(t) + \gamma'(t) y)^2, \quad \text{for } 0 \le t \le T \text{ and } 0 \le y \le 1.$

H3: $k > 0$, and $\eta_1 . \eta_2 > 0$.

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We will now consider a change of variables to transform the domain Q_t in a cylindrical domain Q. Observe that, when (x,t) varies in Q_t the point (y,t) of $I\!R^2$, with $y=(x-\alpha(t))/\gamma(t)$ varies in the cylinder $Q=(0,1)\times(0,T)$. Thus, we have the application, defined by

$$\begin{aligned} \mathcal{T}\,:\,Q_t &\to Q=(0,1)\times]0,T[\\ (x,t) &\mapsto (y,t)=\Big(\frac{x-\alpha(t)}{\gamma(t)}\,,\,t\Big), \end{aligned} \tag{1}$$

belong C^2. The inverse $\mathcal{T}^{-1}$ is also C^2. This technique that transforms the equation from moving boundary in fixing boundary was initially employed by Medeiros at al. in [9] and [10].

Doing the change of variable $v(y,t)=u(\alpha(t)+\gamma(t)y,t)$ and $\phi(y,t)=\theta(\alpha(t)+\gamma(t)y,t)$ and applying in the problema (**I**), we obtain the following problem equivalent, defined in a cylindrical domain:

$$\textbf{(II)}\left[\begin{array}{ll} \dfrac{\partial^2 v}{\partial t^2}-\dfrac{\partial}{\partial y}\Big(a_1(y,t)\dfrac{\partial v}{\partial y}\Big)+a_2(t)\dfrac{\partial\phi}{\partial y}+a_3(y,t)\dfrac{\partial^2 v}{\partial y\partial t}+a_4(y,t)\dfrac{\partial v}{\partial y}=0, & \text{in } Q\\ \dfrac{\partial\phi}{\partial t}-b_1(t)\dfrac{\partial^2\phi}{\partial y^2}+b_2(t)\dfrac{\partial^2 v}{\partial y\partial t}+b_3(y,t)\dfrac{\partial\phi}{\partial y}+b_4(t)\dfrac{\partial v}{\partial y}+b_5(y,t)\dfrac{\partial^2 v}{\partial y^2}=0, & \text{in } Q\\ v=\phi=0; & \forall\,(y,t)\in\Sigma,\\ v(y,0)=v_0(y),\ \dfrac{\partial v}{\partial t}(y,0)=v_1(y),\ \phi(y,0)=\phi_0(y), & \text{for}\quad 0<y<1. \end{array}\right.$$

where

$$\begin{array}{lll} b_1(t)=k/\gamma(t)^2\,, & b_2(t)=\eta_2/\gamma(t)\,, & b_3(y,t)=-(\alpha'(t)+\gamma'(t)y)/\gamma(t)\,,\\ b_4(t)=-\gamma'(t)/\gamma(t)^2\,, & b_5(y,t)=b_3(y,t)/\gamma(t)\,, & a_1(y,t)=1/\gamma(t)^2-\Big(b_3(y,t)\Big)^2\,,\\ a_2(t)=\eta_1/\gamma(t)\,, & a_3(y,t)=2b_3(y,t)\,, & a_4(y,t)=-(\alpha''(t)+\gamma''(t)y)/\gamma(t). \end{array}$$

Let $((\ ,\ ))$, $\|\cdot\|$ and $(\ ,\ )$, $|\cdot|$, be respectively the scalar product and the norms in $H_0^1(0,1)$ and $L^2(0,1)$. We denote by $a_1(t,v,w)$ and $b_1(t,v,w)$ the bilinear forms, continuous, symmetric and coercive, defined in $H_0^1(0,1)$ by

$$a_1(t,v,w)=\int_0^1 a_1(y,t)\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\,dy,\qquad b_1(t,v,w)=\int_0^1 b_1(t)\frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\,dy. \tag{2}$$

2 Existence and Uniqueness

We shall first establish the existence and uniqueness of problem (**II**) as an auxiliary theorem and then prove the original problem (**I**).

Theorem 1 *Under the hypotheses* (H1), (H2) *and* (H3) *and given the initial dates*

$$\{u_0,\ \theta_0\}\in H_0^1(\Omega_0)\cap H^2(\Omega_0),\quad u_1\in H_0^1(\Omega_0),$$

there exists functions $\{u;\ \theta\}:Q_t\to I\!R$, *solution of Problem* (I) *in* Q_t, *satisfying the following conditions:*

1. $u\in L^\infty(0,T;H_0^1(\Omega_t)\cap H^2(\Omega_t))$, $\quad u'\in L^\infty(0,T;H_0^1(\Omega_t))$, $\quad u''\in L^\infty(0,T;L^2(\Omega_t))$,
2. $\theta\in L^2(0,T;H_0^1(\Omega_t)\cap H^2(\Omega_t))$, $\quad \theta'\in L^2(0,T;H_0^1(\Omega_t))$.

Theorem 2 *Under the hypotheses* (H1), (H2) *and* (H3) *and given the initial dates*

$$\{v_0,\ \phi_0\}\in H_0^1(0,1)\cap H^2(0,1),\quad v_1\in H_0^1(0,1),$$

there exists functions $\{v;\ \phi\}:Q\to I\!R$, *solution of Problem* (II) *in* Q , *satisfying the following conditions:*

1. $v\in L^\infty(0,T;H_0^1(0,1)\cap H^2(0,1))$, $\quad v'\in L^\infty(0,T;H_0^1(0,1))$, $\quad v''\in L^\infty(0,T;L^2(0,1))$,
2. $\phi\in L^2(0,T;H_0^1(0,1))\cap H^2(0,1)$, $\quad \phi'\in L^2(0,T;H_0^1(0,1))$.

Proof of Theorem 2. To prove the theorem, we introduce the approximate solutions. Let $T > 0$ and denote by V_m the subspace spanned by $\{w_1, w_2, ..., w_m\}$, where $\{w_\nu,\ \lambda_\nu;\ \nu = 1, \cdots m\}$ are solutions of the spectral problem $((w_i, v)) = \mu(w_i, v),\ \forall v \in H_0^1(0,1)$. If $\{v_m; \phi_m\} \in V_m$ then its can be represented by

$$v_m = \sum_{\nu=1}^{m} d_{\nu m}(t) w_\nu(y), \quad \phi_m = \sum_{\nu=1}^{m} g_{\nu m}(t) w_\nu(y) \tag{3}$$

Let us consider $\{v_m; \phi_m\}$ solutions of the system of ordinary differential equations,

$$\textbf{(III)} \left[\begin{array}{l} (v''_m, w) + a_1(t, v_m, w) + a_2(\dfrac{\partial \phi_m}{\partial y}, w) + (a_3 \dfrac{\partial v'_m}{\partial y}, w) + (a_4 \dfrac{\partial v_m}{\partial y}, w) = 0, \\ (\phi'_m, w) + b_1(t, \phi_m, w) + b_2(\dfrac{\partial v'_m}{\partial y}, w) + (b_3 \dfrac{\partial \phi_m}{\partial y}, w) + (2b_4 \dfrac{\partial v_m}{\partial y}, w) + (b_5 \dfrac{\partial v_m}{\partial y}, \dfrac{\partial w}{\partial y}) = 0, \\ v_m(0) = v_{0m} \to v_0, \quad in \quad H_0^1(0,1) \cap H^2(0,1), \\ v'_m(0) = v_{1m} \to v_1 \quad in \quad H_0^1(0,1), \\ \phi_m(0) = \phi_{0m} \to \phi_0 \quad in \quad H_0^1(0,1) \cap H^2(0,1), \end{array} \right.$$

where $w \in V_m$. The system (III) has local solution in the interval $(0, T_m)$. To extend the local solution to the interval $(0, T)$ independent of m, the following estimates are necessary:

A Priori Estimate

Taking $w = v'_m$ and $w = \phi_m$ in the equation $(\text{III})_1$ and $(\text{III})_2$, respectively, we get

$$\frac{1}{2}\frac{d}{dt}|v'_m|^2 + a_1(t, v_m, v'_m) + a_2\Big(\frac{\partial \phi_m}{\partial y}, v'_m\Big) + \Big(a_3 \frac{\partial v'_m}{\partial y}, v'_m\Big) + \Big(a_4 \frac{\partial v_m}{\partial y}, v'_m\Big) = 0 \tag{4}$$

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\phi_m|^2 + b_1(t, \phi_m, \phi_m) + b_2\Big(\frac{\partial v'_m}{\partial y}, \phi_m\Big) \quad &+ \quad \Big(b_3 \frac{\partial \phi_m}{\partial y}, \phi_m\Big) + 2b_4\Big(\frac{\partial v_m}{\partial y}, \phi_m\Big) + \\ &+ \quad \Big(b_5 \frac{\partial v_m}{\partial y}, \frac{\partial \phi_m}{\partial y}\Big) = 0 \end{aligned} \tag{5}$$

Note that, we have the following equalities and inequality:

$$\begin{aligned} a_1(t, v_m, v'_m) &= \frac{1}{2}\frac{d}{dt}\ a_1(t, v_m, v_m) - \frac{1}{2}\Big(a'_1 \frac{\partial v_m}{\partial y}\ ,\ \frac{\partial w}{\partial y}\Big), \\ a_2\Big(\frac{\partial \phi_m}{\partial y}, v'_m\Big) &= -\frac{\eta_1}{\eta_2} b_2\Big(\frac{\partial v'_m}{\partial y} \phi_m\Big), \\ \Big(a_3 \frac{\partial v'_m}{\partial y}, v'_m\Big) &= -\frac{\gamma'}{\gamma}|v'_m|^2, \\ \Big(b_3 \frac{\partial \phi_m}{\partial y}, \phi_m\Big) &= \frac{\gamma'}{2\gamma}|\phi_m|^2, \\ \Big|(b_5 \frac{\partial v_m}{\partial y}, \frac{\partial \phi_m}{\partial y})\Big| &\leq c\|v_m\|^2 + \frac{1}{2} b_1(t, \phi_m, \phi_m) \end{aligned} \tag{6}$$

Multiplying (4) by (η_2/η_1), adding to (5) and using (6) we have;

$$\begin{aligned} \frac{\eta_2}{2\eta_1}\frac{d}{dt}\ \Big(|v'_m|^2 + a_1(t, v_m, v_m) \quad &+ \quad \frac{\eta_1}{\eta_2}|\phi_m|^2\Big) + b_1(t, \phi_m, \phi_m) \\ &\leq \quad C\Big(|v'_m|^2 + \|v_{0m}\|^2 + |\phi_m|^2\Big) \end{aligned} \tag{7}$$

So, integrating (7), using that $a_1(t,v,w)$ and $b_1(t,v,w)$ are coercive forms and applying the Gronwall's inequality, we get

$$|v_m'|^2 + \|v_m\|^2 + |\phi_m|^2 + \int_0^t \|\phi_m\|^2 \le c_1\Big(|v_{1m}|^2 + \|v_{0m}\|^2 + |\phi_{0m}|^2\Big)e^{c_2 T}. \tag{8}$$

A Second Estimate

Taking the derivative with respect the t, of approximate systems $(\mathrm{III})_{1,2}$, and also $w = v_m''$, $w = \phi_m'$, respectively, we obtain

$$\begin{aligned}(v_m''', v_m'') + a_1(t, v_m', v_m'') &+ a_2\Big(\frac{\partial \phi_m'}{\partial y}, v_m''\Big) + \Big(a_3\frac{\partial v_m''}{\partial y}, v_m''\Big) + \Big((a_3' + a_4)\frac{\partial v_m'}{\partial y}, v_m''\Big) \\ &+ a_1'(t, v_m, v_m'') + \Big(a_2'\frac{\partial \phi_m}{\partial y}, v_m''\Big) + \Big(a_4'\frac{\partial v_m}{\partial y}, v_m''\Big) = 0\end{aligned} \tag{9}$$

and

$$\begin{gathered}(\phi_m'', \phi_m') + b_1(t, \phi_m', \phi_m') + b_2\Big(\frac{\partial v_m''}{\partial y}, \phi_m'\Big) + \Big(b_3\frac{\partial \phi_m'}{\partial y}, \phi_m'\Big) + \\ (2b_4 + b_2')\Big(\frac{\partial v_m'}{\partial y}, \phi_m'\Big) - \Big(b_5\frac{\partial v_m'}{\partial y}, \frac{\partial \phi_m'}{\partial y}\Big) + b_1'(t, \phi_m, \phi_m') + \\ \Big(b_3'\frac{\partial \phi_m}{\partial y}, \phi_m'\Big) + 2b_4'\Big(\frac{\partial v_m}{\partial y}, \phi_m'\Big) + \Big(b_5'\frac{\partial v_m}{\partial y}, \frac{\partial \phi_m'}{\partial y}\Big) = 0,\end{gathered} \tag{10}$$

We also have the following equality and inequality,

$$\begin{aligned}&a_1(t, v_m', v_m'') = \frac{1}{2}\frac{d}{dt}\ a_1'(t, v_m', v_m') - \frac{1}{2}a_1'(t, v_m', v_m') \\ &\Big(a_3\frac{\partial v_m''}{\partial y}, v_m''\Big) = \frac{\gamma'}{\gamma}|v_m''|^2 \\ &a_1'(t, v_m, v_m'') = \frac{d}{dt}\ \Big(a_1'\frac{\partial v_m}{\partial y}, \frac{\partial v_m'}{\partial y}\Big) - \Big(a_1''\frac{\partial v_m}{\partial y}, \frac{\partial v_m'}{\partial y}\Big) - \Big(a_1'\ \frac{\partial v_m'}{\partial y}, \frac{\partial v_m'}{\partial y}\Big) \\ &a_2\Big(\ \frac{\partial \phi_m'}{\partial y}, v_m''\Big) = -\frac{\eta_1.b_2}{\eta_2}\ \Big(\frac{\partial v_m''}{\partial y}, \phi_m'\Big) \\ &\Big(b_3\ \frac{\partial \phi_m'}{\partial y}\ ,\ \phi_m'\Big) = \frac{1}{2}\frac{\gamma'}{\gamma}\ |\phi_m'|^2 \\ &\Big|\Big(a_1'\frac{\partial v_m}{\partial y}, \frac{\partial v_m'}{\partial y}\Big)\Big| \le C\|v_m\|^2 + \frac{\eta_2}{4\eta_1}a_1(t, v_m', v_m'),\end{aligned} \tag{11}$$

Multiplying (9) by (η_1/η_2), adding to (10) and using (11), we obtain

$$\begin{aligned}\frac{\eta_2}{2\eta_1}\frac{d}{dt}\Big\{|v_m''|^2 &+ a_1(t, v_m', v_m') + \Big(a_1'\frac{\partial v_m}{\partial y}, \frac{\partial v_m'}{\partial y}\Big) + \frac{\alpha}{\beta}\ |\phi_m'|\Big\} + b_1(t, \phi_m', \phi_m') \\ &\le C\Big(\|v_m\|^2 + \|v_m'\|^2 + |v_m''|^2 + \|\phi_m\|^2 + |\phi_m'|^2\Big)\end{aligned} \tag{12}$$

But from $(\mathrm{III})_{1,5}$ we have that $|v_m''(0)|^2$ and $|\phi_m'(0)|^2$ are bounded. Hence, integrating (12) with respect a t, and applying the Gronwall's inequality, we get

$$\|v_m'\|^2 + |v_m''|^2 + |\phi_m'|^2 + \int_0^t \|\phi_m'\|^2 \le C \tag{13}$$

A Third Estimate

Taking $w = -\,\partial^2 v_m/\partial y^2$ and $w = -\,\partial^2 \phi_m/\partial y^2$, in the approximate systems $(\mathrm{III})_{1,2}$, we have,

$$\begin{gathered}\Big(v_m'', -\frac{\partial^2 v_m}{\partial y^2}\Big) + a_1\Big(t, v_m, -\frac{\partial^2 v_m}{\partial y^2}\Big) + a_2\Big(\frac{\partial \phi_m}{\partial y}, -\frac{\partial^2 v_m}{\partial y^2}\Big) + \\ \Big(a_3\frac{\partial v_m'}{\partial y}, -\frac{\partial^2 v_m}{\partial y^2}\Big) + \Big(a_4\frac{\partial v_m}{\partial y}, -\frac{\partial^2 v_m}{\partial y^2}\Big) = 0\end{gathered} \tag{14}$$

and

$$\begin{aligned}&\Big(\phi_m', -\frac{\partial^2 \phi_m}{\partial y^2}\Big) + b_1\Big(t, \phi_m, -\frac{\partial^2 \phi_m}{\partial y^2}\Big) + b_2\Big(\frac{\partial v_m'}{\partial y}, -\frac{\partial^2 \phi_m}{\partial y^2}\Big) + \\ &2b_4\Big(\frac{\partial v_m}{\partial y}, -\frac{\partial^2 \phi_m}{\partial y^2}\Big) + \Big(b_5\frac{\partial v_m}{\partial y}, -\frac{\partial^3 \phi_m}{\partial y^3}\Big) = 0\end{aligned} \tag{15}$$

Note that, we have the following equalities:

$$\begin{aligned}a_1\Big(t, v_m, -\frac{\partial^2 v_m}{\partial y^2}\Big) &= a_1\Big(t, \frac{\partial v_m}{\partial y}, \frac{\partial v_m}{\partial y}\Big) + \Big(\frac{\partial a_1}{\partial y}\,\frac{\partial v_m}{\partial y}, \frac{\partial^2 v_m}{\partial y^2}\Big) \\ b_1\Big(t, \phi_m, -\frac{\partial^2 \phi_m}{\partial y^2}\Big) &= b_1\Big(t, \frac{\partial \phi_m}{\partial y}, \frac{\partial \phi_m}{\partial y}\Big) \\ \Big(b_5\frac{\partial v_m}{\partial y}, -\frac{\partial^3 \phi_m}{\partial y^3}\Big) &= \Big(b_5\frac{\partial^2 v_m}{\partial y^2}, \frac{\partial^2 \phi_m}{\partial y^2}\Big) - \Big(\frac{\partial b_5}{\partial y}\,\frac{\partial v_m}{\partial y}, \frac{\partial^2 \phi_m}{\partial y^2}\Big)\end{aligned} \tag{16}$$

From (14), (15 and (16) and since that $a_1(t, v, w)$ and $b_1(t, v, w)$ are coercive forms, we obtain;

$$\Big|\frac{\partial^2 v_m}{\partial y^2}\Big|^2 \le c_6\Big(\|\phi_m\|^2 + |v_m''|^2 + \|v_m'\|^2 + \|v_m\|^2 \tag{17}$$

$$\Big|\frac{\partial^2 \phi_m}{\partial y^2}\Big|^2 \le c_7\Big(|\phi_m'|^2 + \|\phi_m\|^2 + |v_m''|^2 + \|v_m'\|^2 + \|v_m\|^2 \tag{18}$$

The estimates obtained in (8), (13), (17) and (18), permit to pass the limits in the approximate systems $(\mathrm{III})_{1,2}$, to the Galerkin method and then we get the solutions $\{v, \phi\}$ in the sense defined in the Theorem 2.

Uniqueness of Solution

Let $\{\hat{v}, \hat{\phi}\}$ and $\{\widetilde{v}, \widetilde{\phi}\}$ be two solutions of Problem (II). Then $v = \hat{v} - \widetilde{v}$ and $\phi = \hat{\phi} - \widetilde{\phi}$ are also solutions of Problem (II), with initial condition nulls. Then, multiplying the equation $(\mathrm{II})_{1,2}$, respectively by $(\eta_1/\eta_2)v$ and ϕ, we obtain;

$$|v'|^2 + \|v\|^2 + |\phi|^2 \le c\int_0^t \Big(|v'|^2 + \|v\|^2 + |\phi|^2\Big) \tag{19}$$

So, applying Gronwall Lemma, we have $|v'|^2 + \|v\|^2 + |\phi|^2 = 0$ and therefore, we conclude that $v = \phi = 0$ for all $0 < t < T$. This, finishes the proof of Theorem 2. □

The Original Problem (I)

Now let us restate the previous results for the original problem (I) and will prove the Theorem 2.

Proof of Theorem 1. Let $\{v, \phi\}$ be solution of Problem (**II**), with initial data, given by;

$$v_0(y) = u_0\Big(\alpha(0) + \gamma(0)y\Big) \ , \qquad \phi_0(y) = \theta_0\Big(\alpha(0) + \gamma(0)y\Big) \ ,$$
$$v_1(y) = u_1\Big(\alpha(0) + \gamma(0)y\Big) + \Big(\alpha'(0) + \gamma'(0)y\Big)\ u_0'\Big(\alpha(0) + \gamma(0)y\Big)$$

Considerer the functions $u(x,t) = v(y,t)$ and $\theta(x,t) = \phi(y,t)$, where $x = \alpha(t) + \gamma(t)y$. To verify that $u(x,t)$ and $\theta(x,t)$, under the hypotheses of Theorem 1, are a solution of problem (I), it is sufficient to observe that the mapping: $(x,t) \to ((x-\alpha)/\gamma \ , \ t)$ of the domain Q_t into $Q = (0,1) \times (0,T)$ is of class C^2. Since that

1. $\dfrac{\partial u^2}{\partial x^2} = \dfrac{1}{\gamma}\dfrac{\partial v^2}{\partial y^2} \ , \qquad \eta_1 \dfrac{\partial \theta}{\partial x} = a_2 \dfrac{\partial \theta}{\partial y},$
2. $u'' = v'' - \dfrac{\partial}{\partial y}\Big(a_1 \dfrac{\partial v}{\partial y}\Big) + a_3 \dfrac{\partial^2 v}{\partial y \partial t} + a_4 \dfrac{\partial v}{\partial y} + \dfrac{1}{\gamma^2}\dfrac{\partial^2 v}{\partial y^2}$
3. $k\dfrac{\partial^2 \theta}{\partial x^2} = b_1 \dfrac{\partial^2 \phi}{\partial y^2} \ , \qquad \dfrac{\partial \theta}{\partial t} = \dfrac{\partial \phi}{\partial t} + b_3 \dfrac{\partial \phi}{\partial y},$
4. $\eta_2 \dfrac{\partial^2 u}{\partial x \partial t} = b_2 \dfrac{\partial^2 v}{\partial y \partial t} + b_4 \dfrac{\partial v}{\partial y},$

and from problem (II) we also have that $\{u, \theta\}$ satisfy the problem (I).

The regularity of $\{v(y,t), \phi(y,t)\}$ given by Theorem 2 implies that $\{u(x,t), \theta(x,t)\}$ is a solution of problem (I) and the uniqueness of the solution of problem (I) is a direct consequence of the uniqueness of problem (II). □

3 Approximate Solution

Our goal in this section is the numerical implementation of approximate solutions. To obtain the numerical approximate solutions we will use both finite element method and finite difference method. Besides, some numerical experiments will be presented to analyze the effect of the moving boundary in the systems thermoelastics.

For convenience, our numerical analysis using finite elements methods approximation will be based on the equivalent problem (II) in the rectangular domain, in place the problem (I) where the domain is dependent of the time. We also will consider in the numerical simulations the following change in the boundary functions, $\alpha(t) = -K(t)$ and $\beta(t) = K(t)$. Of course, that all the results are still valid.

Note that, now we have

$$Q_t = \Big\{(x,t) \in I\!R^2; \ x = K(t)y, \ y \in (-1,1), \ t \in \ (0,T) \ \Big\} \tag{20}$$

being the non-cylindrical domain with boundary $\Sigma_t = \displaystyle\bigcup_{0<t<T} \{-K(t), K(t)\} \times \{t\}$, and consequently we have the cylindrical domain $Q = (-1,1) \times (0,T)$. This form we obtain the following relation between the functions;

$$u(x,t) = v(y,t) = v\left(\frac{x}{K(t)}, t\right) \quad \text{and} \quad \theta(x,t) = \phi(y,t) = \phi\left(\frac{x}{K(t)}, t\right). \tag{21}$$

3.1 Variational Form of the Problem

Let us consider the following variational form, given by $(\mathbf{III})_{1,2}$

$$(v_m'', w) + a_1(t, v_m, w) + a_2\Big(\frac{\partial \phi_m}{\partial y}, w\Big) + \Big(a_3 \frac{\partial v_m'}{\partial y}, w\Big) + \Big(a_4 \frac{\partial v_m}{\partial y}, w\Big) = 0 \tag{22}$$

and

$$(\phi'_m, w) + b_1(t, \phi_m, w) + b_2\Big(\frac{\partial v'_m}{\partial y}, w\Big) + \Big(b_3\frac{\partial \phi_m}{\partial y}, w\Big) + 2b_4\Big(\frac{\partial v_m}{\partial y}, w\Big) - \Big(b_5\frac{\partial v_m}{\partial y}, \frac{\partial w}{\partial y}\Big) = 0, \quad \forall w \in V_m \tag{23}$$

where now, using (20), the functions b_i and a_i are given by,

$$\begin{array}{lll} b_1 = k/K^2(t), & b_2 = \eta_2/K(t), & b_3 = -K'(t)y/K(t), \\ b_4 = -K'(t)/K^2(t), & b_5 = -b_3/K(t), & a_1 = 1/K^2(t) - b_3^2, \\ a_2 = \eta_1/K(t), & a_3 = 2b_3, & a_4 = -K''(t)y/K(t) \end{array} \tag{24}$$

Galerkin Methods and Approximation

Consider the functions $\{v_m; \phi_m\} \in V_m$ defined in (3). Taking $w = \varphi_j(y)$ and substituting in (22) and (23), we obtain the systems of ordinary equations, given by

$$\begin{cases} A\, d''(t) + \Big(B(t) + E(t)\Big)d(t) + (a_2C)g(t) + D(t)\, d'(t) = 0 \\ A\, g'(t) + \Big(b_1F + G(t)\Big)g(t) + (b_2C)d'(t) + \Big(2b_4C + R(t)\Big)d(t) = 0 \end{cases} \tag{25}$$

where

$$\begin{array}{ll} A = \displaystyle\int_{-1}^{1} \varphi_i(y)\, \varphi_j(y)\, dy, & B(t) = \displaystyle\int_{-1}^{1} a_1 \frac{\partial \varphi_i(y)}{\partial y}\, \frac{\partial \varphi_j(y)}{\partial y}\, dy, \\ C = \displaystyle\int_{-1}^{1} \frac{\partial \varphi_i(y)}{\partial y}\, \varphi_j(y)\, dy, & D(t) = \displaystyle\int_{-1}^{1} a_3 \frac{\partial \varphi_i(y)}{\partial y}\, \varphi_j(y)\, dy, \\ E(t) = \displaystyle\int_{-1}^{1} a_4 \frac{\partial \varphi_i(y)}{\partial y}\, \varphi_j(y)\, dy, & F = \displaystyle\int_{-1}^{1} \frac{\partial \varphi_i(y)}{\partial y}\, \frac{\partial \varphi_j(y)}{\partial y}\, dy, \\ G(t) = \displaystyle\int_{-1}^{1} b_3 \frac{\partial \varphi_i(y)}{\partial y}\, \varphi_j(y)\, dy, & R(t) = \displaystyle\int_{-1}^{1} -b_5 \frac{\partial \varphi_i(y)}{\partial y}\, \frac{\partial \varphi_j(y)}{\partial y}\, dy. \end{array} \tag{26}$$

$d(t) = \Big(d_1(t), \cdots d_m(t)\Big)^t$ and $g(t) = \Big(g_1(t), \cdots g_m(t)\Big)^t$.

For numerical convenient, using (24), we shall rewrite the $B(t)$ matrix in a form $B = B^1 + B^2$, where

$$B^1(t) = \frac{1}{K(t)^2}\int_{-1}^{1} \frac{\partial \varphi_i(y)}{\partial y}\, \frac{\partial \varphi_j(y)}{\partial y}\, dy, \quad B^2(t) = -\Big(\frac{K'(t)}{K(t)}\Big)^2 \int_{-1}^{1} (y)^2\, \frac{\partial \varphi_i(y)}{\partial y}\, \frac{\partial \varphi_j(y)}{\partial y}\, dy$$

3.2 Finite Element Approximation

We now present a semi-discrete formulation for problem (25) using the Galerkin finite element method to discretize the spatial variable. To boundary values not dependent of the time t, if we add (increase) that the boundary values are variables with the time t, then we obtain, like the Problem **(II)**, coefficients a_i and b_i all dependents of the time t and spatial variable y, with more reason, same for regular domains Ω, is quite difficult to solve explicitly. Then some numerical methods may be used to find the solution approximately; the finite element method is just one such method which we now summarize. We first applied the method to obtain the approximation solution of the exact solution $v(y, t)$ of the Problem **(II)** and after, using the transformation (21) we have the approximation solution of the $u(x, t)$ for the Problem **(I)** in the domain Q_t.

First, we divide the domain $\Omega = (0, 1)$ in local domain $\Omega_i = (x_i, x_{i+1})$. Then, $\Omega = int\big(\cup_{i=1}^{m} \bar{\Omega}_i\big)$ and $\Omega_i \cap \Omega_j = \emptyset, \quad if \quad i \neq j$. In finite element methods, the φ_i are piecewise polynomials of some degree in

Ω and vanish on $\partial\Omega$. More specifically, in this work, we have used the hat function, i.e,

$$\varphi_i(y) = \begin{cases} \dfrac{y - y_{i-1}}{h}, & \forall\, y \in [y_{i-1}, y_i] \\ \dfrac{y_{i+1} - y}{h}, & \forall\, y \in [y_i, y_{i+1}] \\ 0, & \forall\, y \notin [y_{i-1}, y_{i+1}] \end{cases} \tag{27}$$

where we are consider the uniform mesh, $h = h_i = y_{i+1} - y_i$, $i = 1, 2, \ldots, m$ in the discretization in m-parts, with $-1 = y_1 < y_2 < \cdots < y_{m+1} = 1$. Note that, if $|i - j| > 2$, then $(\varphi_i, \varphi_j) = 0$, and $(\partial\varphi_i/\partial y, \partial\varphi_j/\partial y) = 0$. Hence all the matrix of systems are tridiagonal.

Matrix Calculation

For each Ω_i, we have to calculate each integrals defined in (26), using the functions (24), (27) and its derived. Doing the calculus, we obtain, respectively the following elements, for each tridiagonal matrix A, B^1, B^2, C, D, E, F, G and R:

$$\begin{aligned}
& a_{ii} = \frac{4}{3m}\ , \quad a_{i,\, i+1} = a_{\,i+1,\, i} = \frac{1}{3m}\ , \\
& b^1_{ii} = \frac{m}{K^2}\ , \quad b^1_{i,\, i+1} = b^1_{i+1,\, i} = -\frac{m}{2K^2}\ , \\
& b^2_{ii} = -\frac{m(K')^2}{3K^2}\left(3y_i^2 + \frac{4}{m^2}\right),\ b^2_{i,\, i+1} = b^2_{i+1,\, i} = \frac{m\,(K')^2}{6K^2}\left(3y_i^2 + \frac{6y_i}{m} + \frac{4}{m^2}\right) \\
& c_{ii} = 0, \quad c_{i,\, i+1} = -\frac{1}{2}, \quad c_{\,i+1,\, i} = \frac{1}{2} \\
& d_{ii} = \frac{4K'}{3mK},\ d_{\,i,\, i+1} = \frac{K'}{3K}\left(\frac{4}{m} + 3y_i\right),\ d_{\,i+1,\, i} = -\frac{K'}{3K}\left(\frac{2}{m} + 3y_i\right), \\
& e_{ii} = \frac{2K''}{3mK},\ e_{i,\, i+1} = \frac{K''}{6K}\left(\frac{4}{m} + 3y_i\right),\ e_{i+1,i} = -\frac{K''}{6K}\left(\frac{2}{m} + 3y_i\right), \\
& f_{ii} = m, \quad f_{i,\, i+1} = f_{\,i+1,i} = -\frac{m}{2} \\
& g_{ii} = \frac{2K'}{3m\ K},\ g_{i,\, i+1} = \frac{K'}{6K}\left(\frac{4}{m} + 3y_i\right),\ g_{i+1,i} = -\frac{K'}{6K}\left(\frac{2}{m} + 3y_i\right), \\
& r_{i,\, i} = -\frac{mK'y_i}{K^2},\ r_{i,\, i+1} = r_{i+1,\, i} = \frac{K'}{2K^2}\left(1 + my_i\right),
\end{aligned} \tag{28}$$

3.3 Finite Difference Method

The equation (25) represent a system of ordinary differential equations of second order and due to matrices characteristics (dependence of the variables y and t) of system, obtaining the solution is not always possible. So, we will apply a numerical method to obtain the approximated solution for the system (25), using the approximate Newmark's method (see, for instance, Hugles [6], pp 493).

Let $d^n = d(t_n)$ and $g^n = g(t_n)$ be the approximate solution of the exact solution $d(t)$ and $g(t)$ of $(25)_{1,2}$, respectively, where we denote the discrete times in the interval $[0, T]$ by $t_n = n\Delta t, \quad n = 0, 1 \cdots N$.

To $\delta \geq 1/4$, with $\delta \in I\!R$, consider the following approximation

$$\begin{aligned} d^{*n} &= \delta\ d^{n+1} + (1-2\delta)d^n + \delta\ d^{n-1} \\ g^{*n} &= \delta\ g^{n+1} + (1-2\delta)g^n + \delta g^{n-1}, \end{aligned} \tag{29}$$

and for the first and second derivative, we take the difference operator in the following form

$$\tau d^n = \frac{d^{n+1} - d^{n-1}}{2\Delta t}, \quad \tau g^n = \frac{g^{n+1} - g^{n-1}}{2\Delta t}, \quad \delta^2 d^n = \frac{d^{n+1} - 2d^n + d^{n-1}}{\Delta t^2} \tag{30}$$

which, for this approximate, the discrete error can be showed of order $o(\Delta t^2)$.

Coupled Systems

Let the systems $(25)_{1,2}$ at the discrete mesh points $t_n = n\Delta t$, and substituting the approximation (29) and (30), we obtain the following coupled systems;

$$\left\{ \begin{aligned} &\widehat{A}^n\ d^{n+1} + \widehat{B}^n\ g^{n+1} = \widehat{C}d^n - \widehat{D}d^{n-1} - \widehat{E}g^n - \widehat{F}g^{n-1} \\ &\widetilde{A}^n\ d^{n+1} + \widetilde{B}^n\ g^{n+1} = -\widetilde{C}d^n + \widetilde{D}d^{n-1} - \widetilde{E}g^n + \widetilde{F}g^{n-1} \end{aligned} \right. \tag{31}$$

where, we are denoting

$$\begin{aligned} &\widehat{A}^n = A + \delta\Delta t^2 (B^1)^n + \frac{\Delta t}{2}\, D^n, \quad \widehat{B}^n = a_2^n\ \delta\Delta t^2\ C \\ &\widehat{C}^n = 2A - \Delta t^2\Big((1-2\delta)\ (B^1)^n + (B^2)^n + E^n\Big) \\ &\widehat{D}^n = A + \delta\Delta t^2 (B^1)^n - \frac{\Delta t}{2} D^n, \quad \widehat{E}^n = a_2^n\ (1-2\delta)\Delta t^2 C \\ &\widehat{F}^n = a_2^n\ \delta\Delta t^2 C\,, \quad \widetilde{A}^n = \frac{b_2^n}{2}\ C, \quad \widetilde{B}^n = \frac{A}{2} + b_1^n\ \delta\Delta t\ F \\ &\widetilde{C}^n = \Delta t\Big(2b_4^n\ C + R^n\Big), \quad \widetilde{D}^n = \frac{b_2^n}{2}\ C \\ &\widetilde{E}^n = \Delta t\Big(b_1^n (1-2\delta)\ F + G^n\Big), \quad \widetilde{F}^n = \frac{A}{2} - b_1^n\ \delta\Delta t\ F \end{aligned} \tag{32}$$

To determine the solution $\{d^n, g^n\}$, the coupled system of algebraic equations (31), may be solved by iteration, as follows: To start the iteration, we first taking $n = 0$ in (31) and rewrite the system as

$$\left\{ \begin{aligned} &\widehat{A}^0\ d^1 + \widehat{B}^0\ g^1 = \widehat{C}^0\ d^0 - \widehat{D}^0\ d^{-1} - \widehat{E}^0\ g^0 - \widehat{F}^0\ g^{-1} \\ &\widetilde{A}^0\ d^1 + \widetilde{B}^0\ g^1 = -\widetilde{C}^0\ d^0 - \widetilde{D}^0\ d^{-1} - \widetilde{E}^0\ g^0 + \widetilde{F}^0\ g^{-1} \end{aligned} \right.$$

where the right-hand side is determined by the (starting) values, since the exact solutions $\{v(y,t); \phi(y,t)\}$ are known at time $t = 0$ and $\{v^0; \phi^0\}$ are just the initial values, i.e, $d^0 = v^0(.) = v(.,0)$, $\quad g^0 = \phi^0(.) = \phi(.,0)$, where, we have used (3) and (27).

We can determine a close approximation $\{d^{-1}; g^{-1}\}$ by the second order Taylor extrapolation of $\{v(.,t); \phi(.,t)\}$ from $t^0 = 0$, *viz*;

$$d^{-1} = d^0 - \Delta t\, d'(0) + \frac{\Delta t^2}{2}\, d''(0), \qquad g^{-1} = g^0 - \Delta t\, g'(0) \tag{33}$$

that by the initial condition we have,

$$d'(0) = \frac{\partial v}{\partial t}\,(y_i, 0) = v_1(y_i)$$

and the values,

$$d''(0) = \frac{\partial^2 v}{\partial t^2}(y_i, 0), \quad g'(0) = \frac{\partial \phi}{\partial t}(y_i, 0),$$

are calculated from the field equation $(\mathbf{II})_\mathbf{1}$ and $(\mathbf{II})_\mathbf{2}$, at $t^0 = 0$ and the initial values $v^0(.) = v(., 0) \quad \phi^0(.) = \phi(., 0)$.

The system may be solved uniquely for $\{d^1, g^1\}$, since its coefficient matrix is non singular. Having determined the values $\{d^1, g^1\}$, then for $n = 1, 2, \cdots N$, we obtain the approximate solution $\{d^{n+1}, g^{n+1}\}$ for the coupled system of algebraic equations (31), which can be rewrite in the block matricial form

$$\begin{bmatrix} \widehat{A}^n & \widehat{B}^n \\ \\ \widetilde{A}^n & \widetilde{B}^n \end{bmatrix} \begin{bmatrix} d^{n+1} \\ \\ g^{n+1} \end{bmatrix} = \begin{bmatrix} S^n \\ \\ T^n \end{bmatrix} \tag{34}$$

when the time varies discretely over the interval $[0, T]$, where

$$S^n = \widehat{C} d^n - \widehat{D} d^{n-1} - \widehat{E} g^n - \widehat{F} g^{n-1} \quad \text{and} \quad T^n = -\widetilde{C} d^n + \widetilde{D} d^{n-1} - \widetilde{E} g^n + \widetilde{F} g^{n-1}.$$

The system (34) may be solved uniquely, since the matrix is non-singular. In order to solve the systems we can use de Gauss Elimination, LU factorization, as same as [5, 8] or Uzwa method [5].

Note that each square matrix of linear system have $(m-1)$ ordem, since that all matrix defined by (32) are $(m-1)$ ordem. So the linear systems have $2(m-1) \times 2(m-1)$, with the block matrix and right-hand known by before iteration.

Uncoupled Systems

Since $\{d^n, g^n\}$ must be solved jointly at each time step, the preceding numerical scheme is computationally coupled. From the numerical standpoint the coupled system is larger and hence harder to solve than an uncoupled system involving only d^{n+1} or only g^n at each time step t_n. In order to get uncoupled systems, see [1] and [11], we replace the central difference by the backward extrapolation for the first derivate,

$$d'(t_n) = \frac{1}{2\Delta t}\Big(3d^n - 4d^{n-1} + d^{n-2}\Big) \tag{35}$$

then substituting in the systems $(25)_{1,2}$ together with (29) and (30), we obtain, after some simple calculation,

$$\begin{aligned} \widehat{A}^n\, d^{n+1} &= \widehat{B}^n\, d^n - \widehat{C}^n\, d^{n-1} - \widehat{D}^n\, g^{n+1} - \widehat{E}^n\, g^n - \widehat{F}^n\, g^{n-1} \\ \widetilde{A}^n\, g^{n+1} &= -\widetilde{B}^n\, d^n + \widetilde{C}^n\, d^{n-1} - \widetilde{D}^n\, d^{n-2} - \widetilde{E}^n\, g^n + \widetilde{F}^n\, g^{n-1} \end{aligned} \tag{36}$$

where

$$\begin{aligned} \widehat{A}^n &= A + \delta\Delta t^2 B_1^n + \frac{\Delta t}{2} D^n, & \widehat{B}^n &= 2A - \Delta t^2\Big((1-2\delta)B_1^n + B_2^n + E^n\Big), \\ \widehat{C}^n &= A + \delta\Delta t^2 B_1^n - \frac{\Delta t}{2} D^n, & \widehat{D}^n &= \delta a_2^n \Delta t^2 C, \\ \widehat{E}^n &= (1-2\delta) a_2^n \Delta t^2\, C, & \widehat{F}^n &= \delta a_2^n \Delta t^2 C, \\ \widetilde{A}^n &= \frac{1}{2}\, A + \delta b_1^n \Delta t F, & \widetilde{B}^n &= \frac{1}{2}\Big(3b_2^n + 4b_4^n \Delta t\Big)\, C + \Delta t R^n, \\ \widetilde{C}^n &= 2b_2^n\, C, & \widetilde{D}^n &= \frac{1}{4}\, \widetilde{C}^n, \\ \widetilde{E}^n &= \Delta t\Big((1-2\delta)\, b_1^n F + G^n\Big), & \widetilde{F}^n &= \frac{1}{2}\, A - \delta b_1^n \Delta t F. \end{aligned} \tag{37}$$

To start the iteration, we first taking $n = 0$ in $(36)_2$ and after $n = 0$ in $(36)_1$, so we have that

$$\begin{aligned} \widetilde{A}^0 g^1 &= -\widetilde{B}^0 d^0 + \widetilde{C}^0 d^{-1} - \widetilde{D}^0 d^{-2} - \widetilde{E}^0 g^0 + \widetilde{F}^0 g^{-1} \\ \widehat{A}^0 d^1 &= \widehat{B}^0 d^0 - \widehat{C}^0 d^{-1} - \widehat{D}^0 g^1 - \widehat{E}^0 g^0 - \widehat{F}^0 g^{-1} \end{aligned} \tag{38}$$

The terms of right-hand side of the $(38)_1$, now involve the values $\left\{d^0, d^{-1}, d^{-2}, g^0, g^{-1}\right\}$ are known by (33) a least the term d^{-2}, that can be calculated, taking $n = 0$ in (35);

$$d^{-2} = -3d^0 + 4d^{-1} + 2\Delta t\, d'(0) \tag{39}$$

So, the value of g^1 is calculated in $(38)_1$ and substituting in $(38)_2$ we obtain the value of d^1. Having determined the values d^1, then taking $n = 1$ in $(36)_2$ we obtain the value g^2 and taking $n = 1$ in $(36)_1$, we obtain de value of d^2. Then the numerical scheme may be moved forward alternately between $(36)_2$ and $(36)_1$. This numerical system is computationally uncoupled, since at each step t_n, we have two separate systems. Following the preceding numerical scheme, we obtain the values $\{g^{n+1}, d^{n+1}\}$ for $n = 2, 3 \cdots, N$. These values together with the starting values, constitute the finite element approximate solution to the initial boundary value problem based on Problem **(II)**

4 Numerical Simulation

A numerical example will be given to illustrate some features of the present model, using the method developed for the uncoupled systems that is faster. In the example, we need to determine the constants η_1, η_2 and k, which give rise to the coupling of the parabolic and hyperbolic equation in the thermoelastic system **(I)**. The constants are given by the following formulas

$$\eta_1 = \frac{\overline{\alpha}(3\lambda + 2\mu)\sqrt{\theta_0}}{\sqrt{(\lambda + 2\mu)\rho\, c\, l^2}}, \quad \eta_2 = l\eta_1, \quad k = \frac{\overline{k}}{c\, l\sqrt{\rho(\lambda + 2\mu)}}.$$

where c the is specific heat; $\overline{\alpha}$ the is coefficient of thermal expansion; $\overline{k}$ the is thermal conductivity; $l = 2K(0)$ the is the length of the string; ρ the is density of the string; θ_0 the is initial temperature; λ and μ are the coefficients of Lamé.

For the numerical example, these values will be calculated from the physical properties of *aluminum*. In this case, we have $\mu = 26.24 \times 10^9$ and $\lambda = 58.41 \times 10^9$. Using the thermal and mechanics properties from aluminum, we obtain the approximate values $\eta_1 = 0.164$, $\eta_2 = 0.161$ and $k = 0.177$.

Let us considerer in (29) the weight $\delta = 0.5$, $\Omega = (-K(t), K(t))$ divided in m subintervals, i.e, $h = 2/m$ and $\Delta t = T/N$, for different values of N and T for the discrete time. To calculate the coefficients defined in (24) step by step, the function $K(t)$ that defines the time dependence of the boundary for the non-cylindrical domain Q_t in (20) must be given. In this example it is given by $K(t) = 1 - 1/\exp(t+1)$. Note that, in this case, Q_t tends to Q rapidly as t increases. This particular function was taken in order to satisfy the hypothesis **H2**, i.e, $K'(t) \approx 1$. From the physical point of view, we require that the speed of the end points be less than the "characteristic" speed of the system. When we consider only a wave equation for small vibrations of elastic string or beam equation, both with moving boundaries, are not require the monotonicity of those functions, see [2] and [7].

We consider the initial temperature, the initial position and velocity given by

$$\phi_0(y) = 0.033(1 - y^2), \quad v_0(y) = 0.057(y^2 - 1) \quad \text{and} \quad v_1(y) = 0. \tag{40}$$

In all the figures we are considering the change of variables $y = (x - \alpha(t))/\gamma(t))$.

To obtain the figures Fig.1 - Fig.4, we have used $\Delta t = 0.03$ and $h = 0.02$, with $N = m = 100$ and $T = 3$. Fig.1 and Fig.2, respectively, shows the temperature $\theta(x,t)$ and the displacement $u(x,t)$ in the midpoint $x = 0$.

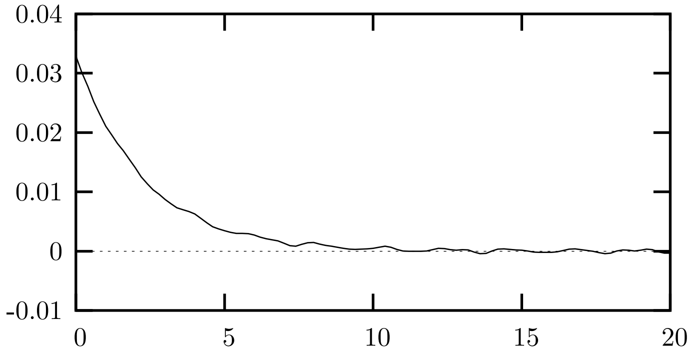

Fig.1: Temperature at midpoint $\theta(0,t)$

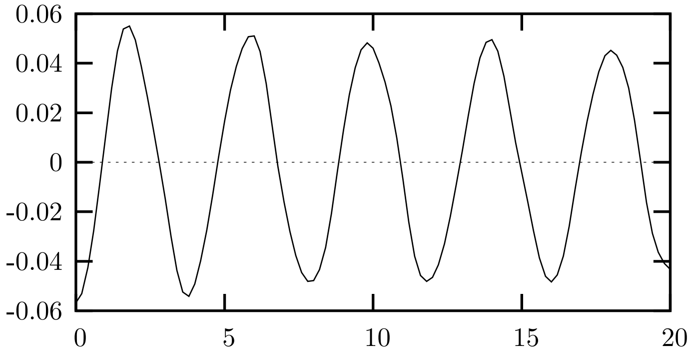

Fig.2: Displacement at midpoint $u(0,t)$

Fig.3 and Fig.4, shows the approximate solution $\theta(x,t^*)$ and $u(x,t^*)$, in the interval $[0,T]=[0,3]$ for different values of t^*, $t^*=0,\ 0.25,\ 0.75,\ 1.5,\ 2.25,\ 3.0$. In these figures, we can see that the maximum temperature and displacement decrease as time increases.

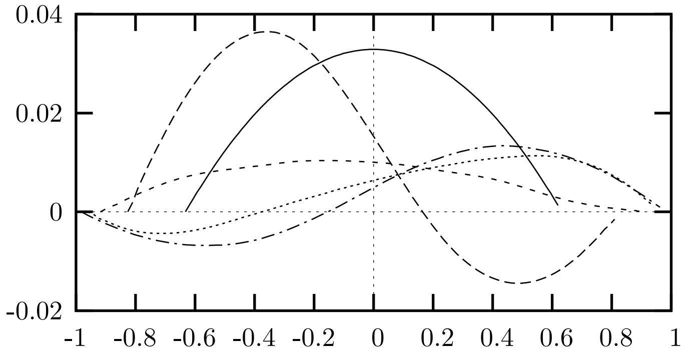

Fig.3: $\theta(x,t^*)$ at $t^*=0,0.25,0.75,1.5,2.25,3.0$

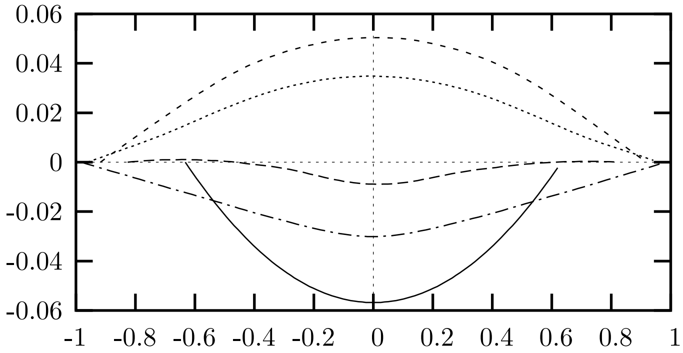

Fig.4: $u(x, t^*)$ at $t^* = 0, 0.25, 0.75, 1.5, 2.25, 3.0$

Note that the interval of the boundary has varied from $[-0.63,\ 0.63]$ to $[-0.98,\ 0.98]$.

To obtain Fig.5 and Fig.6, we have used $\Delta t = 0.125$ and $h = 0.05$. In Fig.5 and Fig.6 the evolution of the displacement function $u(x, t)$ and the evolution of the temperature function $\theta(x, t)$ are plotted, showing the profile of the displacement and temperature, where the time varies from 0 to 8.

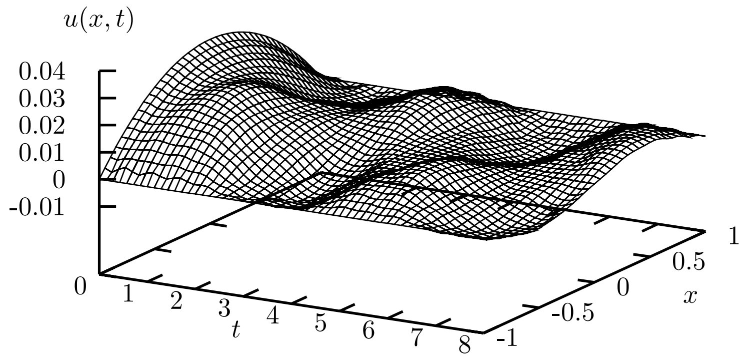

Fig.5: Displacement with 80 steps of time

In Fig.5, we show a continuous profile of the displacement in time, which show clearly the amplitude of the displacement decreases. Similar results are valid also for the temperature profile.

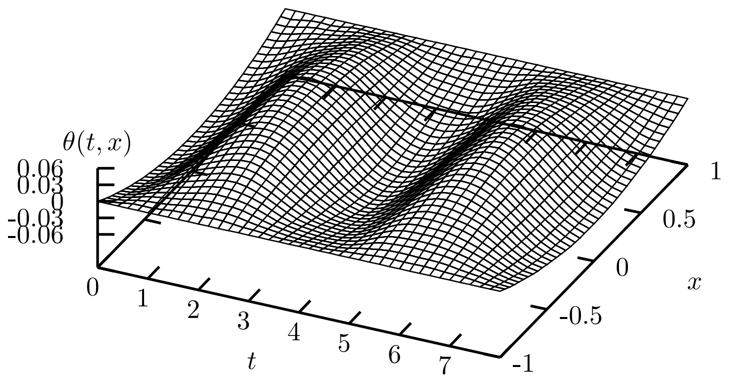

Fig.6: Temperature with 80 steps of time

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